ELASTODYNAMIC GRIFFITH FRACTURE ON PRESCRIBED CRACK PATHS WITH KINKS

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Abstract. We prove an existence result for a model of dynamic fracture based on Griffith's criterion in the case of a prescribed crack path with a kink.

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1. Introduction

In our previous papers [8] and [9] we introduced a model for dynamic fracture based on the following rules:

(a) the displacement satisfies the system of elasto-dynamics out of the time-dependent crack with suitable initial and boundary conditions;
(b) the crack and the displacement satisfy the dynamic energy-dissipation balance: the sum of the kinetic and elastic energy at time t plus the energy spent to produce the crack in the interval $[0,t]$ equals the sum of the kinetic and elastic energy at time 0 plus the work done by the external loads in the interval $[0,t]$;
(c) a maximal dissipation condition holds, which forces the crack to grow whenever it is possible to do so satisfying (a) and (b).

Condition (b) is the dynamic form of Griffith's criterion, originally introduced in [14] in the quasistatic case, and extended to the dynamic case in [18]. Condition (c), proposed in [16], is important because, for given initial and boundary conditions, (a) and (b) are always satisfied by every time-independent crack, together with the corresponding time-dependent displacement.

In [8] we proved the existence of a dynamic crack evolution satisfying (a), (b), and (c) in the antiplane case with a prescribed smooth crack path. This result has been extended in [9] to the case of planar elasticity without a prescribed crack path, assuming a priori regularity constraints on the shape and on the time dependence of the cracks. A critical tool in the proof of these existence results is the continuous dependence on the cracks of the solutions of the system of elasto-dynamics. So far this has been proved only under very strong regularity assumptions on the cracks, including assumptions on the crack speed (see [10] and [6]).

In this paper we study the same problem in the case of prescribed crack paths with kinks, which occurs in many experimental situations (see [13, 15, 3]). Only a few particular solutions of problem (a) are known for a kink (see ([4, 12, 5, 2, 19, 1]). The main difficulty in the proof of the existence of a dynamic evolution satisfying (a), (b), and (c), is that the presence of the kink prevents the use of the known continuous dependence results, at least in the time intervals in which the crack tip is at the kink. We will describe these issues in a little more detail shortly.

To avoid unnecessary technicalities, in this paper we consider only the antiplane case with no external loads. Since the problem can be localized, we assume also that the prescribed crack path Γ starts from the boundary of the reference configuration and has only one kink. The main result is that, for any initial data satisfying some natural assumptions, there exist a time-dependent crack, contained in Γ, and a corresponding time-dependent displacement, such that (a), (b), and (c) are satisfied, where the maximality in (c) is intended only among all competing cracks contained in Γ.

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We now briefly describe the difficulties posed by the presence of a kink in the crack path. The classical approach has largely been based on the idea that for a smoothly growing crack, it is possible to map the changing domain where the elasto-dynamic equation is satisfied to a fixed domain, typically the domain at the initial time. The cost of this approach is the introduction of complications in the wave equation that depend on the regularity of the map. In particular, the new wave equation remains hyperbolic only if the crack speed (and therefore, a certain time derivative of the map) is strictly smaller than the wave speed. This is the reason that we will assume an upper bound c_0 on the crack speed, with $c_0 < 1$, since the wave speed we consider here is 1.

In addition, for our maximal dissipation condition, we need to take limits of maximizing sequences and show the limit satisfies the wave equation, energy balance, and our maximal dissipation definition. This last point is very delicate, and still requires our definition to be weaker than we would prefer. The energy balance is the next most delicate point, and until now required the property that if a sequence of regular crack paths converges strongly enough to a regular limit path, then the corresponding sequence of elasto-dynamic solutions converges strongly to the elasto-dynamic solution corresponding to the regular limit path, proved in [10].

Here, we show that this analysis can be extended to a crack path that is smooth, except for a kink. The analysis before the crack reaches the kink is the same as for a regular crack, and the analysis after the crack tip has passed the kink is also the same as for a regular crack, since the evolving domain after the kink can be mapped smoothly to a fixed one that is also after the kink. The main difficulties are when the crack approaches the kink (Section 6), and when the crack leaves the kink (Section 8). The latter is more serious, as there is no fixed domain that the cracked domain can be smoothly mapped back to, for all times after the kink (or all times shortly after the crack leaves the kink). Section 7 deals with convergence of the maximizing sequence during the time interval at which the crack tip is at the kink.

Our results, together with those of [9], could be easily adapted to the case of planar elasticity with a prescribed crack path with a kink.

2. The model

2.1. **The cracks.** Throughout the paper the reference configuration Ω is a bounded open set in $\mathbb{R}^2$ containing 0. Let $L_1 > 0$, let $L_2 > 0$, and let $\gamma : [-L_1, L_2] \to \overline{\Omega}$ be a curve such that $\gamma(-L_1) \in \partial\Omega$, $\gamma(0) = 0$, and $\gamma(s) \in \Omega$ for every $-L_1 < s < L_2$. The set $\Gamma := \gamma([-L_1, L_2])$ is the prescribed crack path in our model. We assume that $\gamma|_{[-L_1,0]}$ and $\gamma|_{[0,L_2]}$ are of class $C^{3,1}$, i.e., they are of class C^3 and their third derivatives are Lipschitz continuous. We also assume that γ is injective and that $|\dot\gamma(s)| = 1$ for every $s \neq 0$, so that γ is an arc-length parametrization of Γ.

In our model the time-dependent cracks, defined on some time interval $[T_0, T_1]$, have the form

$$K(t) := \gamma([-L_1, \sigma(t)]) , \tag{2.1}$$

where

($\Sigma 1$) $\sigma\colon [T_0, T_1] \to (-L_1, L_2)$ is continuous and nondecreasing.

Given σ satisfying ($\Sigma 1$), we define

$$\tau_1^\sigma := \sup\{t \in [T_0, T_1] : \sigma(t) < 0\} , \tag{2.2}$$

$$\tau_2^\sigma := \inf\{t \in [T_0, T_1] : \sigma(t) > 0\} , \tag{2.3}$$

with the convention $\sup \emptyset := T_0$ and $\inf \emptyset := T_1$. Note that $\tau_1^\sigma \le \tau_2^\sigma$ and that

$$\sigma(t) = 0 \quad \text{for every } t \in [\tau_1^\sigma, \tau_2^\sigma] \tag{2.4}$$

if $\tau_1^\sigma < \tau_2^\sigma$ or $T_0 < \tau_1^\sigma = \tau_2^\sigma < T_1$.

We also need a uniform regularity assumption on σ. In order to apply a continuous dependence result for the solution of the wave equation, throughout the paper we fix $M > 0$ and $c_0 \in (0, 1)$, and we often consider the following assumption:

($\Sigma 2$) the function σ satisfies

$$\sigma|_{[T_0,\tau_1^\sigma]} \in C^{3,1}([T_0, \tau_1^\sigma]) \text{ and } \sigma|_{[\tau_2^\sigma,T_1]} \in C^{3,1}([\tau_2^\sigma, T_1]) , \tag{2.5}$$

$$|\dot\sigma(t)| \le c_0 \quad \text{for every } t \in [T_0, \tau_1^\sigma) \cup (\tau_2^\sigma, T_1] , \tag{2.6}$$

$$\|\sigma|_{[T_0,\tau_1^\sigma]}\|_{C^{3,1}([T_0,\tau_1^\sigma])} \le M \text{ and } \|\sigma|_{[\tau_2^\sigma,T_1]}\|_{C^{3,1}([\tau_2^\sigma,T_1])} \le M , \tag{2.7}$$

where $\|\cdot\|_{C^{3,1}}$ is the sum of the C^3-norm and of the Lipschitz constant of the third derivative.

We fix $T > 0$ and study the dynamic evolution problem on the time interval $[0, T]$. To avoid complete fragmentation we assume that $\sigma(0)$ satisfies $\sigma(0) + c_0 T < L_2$. By (2.6) this implies that $\sigma(t) < L_2$ for every $t \in [0, T]$. Therefore, if Ω is connected, the same property holds for $\Omega \setminus K(t)$ for every $t \in [0, T]$, where $K(t)$ is defined by (2.1).

For technical reasons it is convenient to weaken the regularity assumption (2.5) and to assume only that

($\Sigma 3$) $\sigma\colon [0, T] \to (-L_1, L_2)$ is continuous and there exists a subdivision $0 = T_0 < T_1 < \cdots < T_k = T$ such that $\sigma|_{[T_{j-1}, T_j]}$ satisfies ($\Sigma 2$) on $[T_{j-1}, T_j]$.

We define $\mathrm{sing}(\sigma)$ as the minimal set $\{T_1, \ldots, T_{k-1}\}$ for which ($\Sigma 3$) holds.

Remark 2.1. Note that τ_1^σ and τ_2^σ are intentionally excluded from $\mathrm{sing}(\sigma)$, by the minimality of the set of partition points, and the fact that ($\Sigma 2$) requires regularity only up to τ_1^σ and subsequent to τ_2^σ. This is important because, in our definition of maximal dissipation (Definition 2.5), competitor crack evolutions will be required to match singular points, besides when the kink occurs, which allows the model to predict the kink time.

2.2. **Wave equation on cracking domains.** We now recall the definition of a solution of the wave equation on a time-dependent cracking domain $\Omega \setminus K(t)$, $t \in [0, T]$.

Definition 2.2. Let $T_0 < T_1$ and let $K(t)$ be defined by (2.1), with σ satisfying ($\Sigma 1$). We set $H = L^2(\Omega)$, $V = H^1(\Omega \setminus \Gamma)$, $V_t = H^1(\Omega \setminus K(t))$ for $T_0 \le t \le T_1$, and we define $\mathcal{V}_{T_0}^{T_1}$ to be the space of functions $u \in L^2((0, T); V) \cap H^1((0, T); H)$ such that $u(t) \in V_t$ for a.e. $t \in (T_0, T_1)$.

Definition 2.3. Let $T_0 < T_1$ and let $K(t)$ be defined by (2.1), with σ satisfying ($\Sigma 1$). We say that u is a weak solution of the wave equation on the time-dependent cracking domain $\Omega \setminus K(t)$, $t \in [T_0, T_1]$, with homogeneous Neumann boundary conditon, if $u \in \mathcal{V}_{T_0}^{T_1}$ and

$$-\int_{T_0}^{T_1} (\dot{u}(t), \dot{\varphi}(t))dt + \int_{T_0}^{T_1} (\nabla u(t), \nabla \varphi(t))dt = 0 \tag{2.8}$$

for every $\varphi \in \mathcal{V}_{T_0}^{T_1}$ with $\varphi(T_0) = \varphi(T_1) = 0$.

The existence of a solution with prescribed initial conditions is proved in [7, Theorem 4.2], with a different but equivalent definition, and in [11, Theorem 3.1], with the present definition. Moreover, if u is a solution with $\nabla u \in L^\infty((T_0, T_1); L^2(\Omega \setminus \Gamma; \mathbb{R}^2))$ and $\dot{u} \in L^\infty((T_0, T_1); L^2(\Omega))$, then $u \in C_w([T_0, T_1]; V)$ and $\dot{u} \in C_w([T_0, T_1], H)$, where C_w denotes the space of continuous functions in the weak topology (see [11, Theorem 2.17 and Proposition 2.18]).

It is easy to see that every solution u according to Definition 2.3 is also a solution, in the usual sense, of the wave equation on $[T_0, T_1] \times (\Omega \setminus \Gamma)$ with homogeneous Neumann boundary condition on $\partial\Omega \setminus \Gamma$. Of course the Neumann boundary condition is not satisfied on the whole of Γ, but only on $K(t)$, while on the rest of Γ a suitable transmission condition is satisfied in order to obtain $u(t) \in H^1(\Omega \setminus K(t))$ for a.e. $t \in (T_0, T_1)$.

In our model, given σ satisfying ($\Sigma 1$) and the corresponding crack K defined by (2.1), the time-dependent displacements u satisfy the following assumptions:

(U1) $t \mapsto u(t)$ is a solution of the wave equation on the time-dependent domains $\Omega \setminus K(t)$, $t \in [T_0, T_1]$, with homogeneous Neumann boundary conditon, according to Definition 2.3;

(U2) the energy-dissipation balance is satisfied on $[T_0, T_1]$, i.e., for every $t \in [T_0, T_1]$ we have

$$\frac{1}{2}\|\dot{u}(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega\setminus\Gamma;\mathbb{R}^2)} + \sigma(t) = \frac{1}{2}\|\dot{u}(T_0)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u(T_0)\|^2_{L^2(\Omega\setminus\Gamma;\mathbb{R}^2)} + \sigma(T_0)\,. \tag{2.9}$$

2.3. **Maximal dissipation crack evolutions.** The following definition introduces a class of pairs (σ, u) that will be used in our analysis of the dynamic fracture problem.

Definition 2.4. The pairs (σ, u) which satisfy conditions ($\Sigma 1$), ($\Sigma 3$), (U1), and (U2) on $[0, T]$ are called admissible dynamic evolutions and their collection is denoted by $\mathcal{A}$.

Among all admissible dynamic evolutions our model considers as solutions of the dynamic crack problem only those that satisfy the maximal dissipation condition described in detail by the following definition. For the motivation of this condition we refer to [16, 8].

Definition 2.5. Given $\eta > 0$ we say that $(\sigma, u) \in \mathcal{A}$ satisfies the η-maximal dissipation condition if the following is impossible: there exist $0 \leq \tau_0 < \tau_1 \leq T$ and an alternative admissible dynamic evolution $(\widehat{\sigma}, \widehat{u}) \in \mathcal{A}$ with $\mathrm{sing}(\widehat{\sigma}) \subset \mathrm{sing}(\sigma)$ such that

(a) $(\widehat{\sigma}(t), \widehat{u}(t)) = (\sigma(t), u(t))$ for $0 \leq t \leq \tau_0$
(b) $\widehat{\sigma}(t) \geq \sigma(t)$ for $\tau_0 < t \leq \tau_1$, with strict inequality on a sequence converging to τ_0
(c) $\widehat{\sigma}(\tau_1) > \sigma(\tau_1) + \eta$.

The purpose of this paper is to prove that, given $\eta > 0$, for every initial data (s^0, u^0, u^1) satisfying some natural assumptions there exists an admissible dynamic evolution $t \mapsto (\sigma(t), u(t))$ satisfying the initial condition $(\sigma(0), u(0), \dot{u}(0)) = (s^0, u^0, u^1)$ and the η-maximal dissipation condition.

3. Main result

We are now in a position to state the main result of this paper.

Theorem 3.1. *Let $\eta > 0$, let $K^0 := \gamma([-L_1, s^0])$ with $s^0 \in (-L_1, L_2)$, let $u^0 \in H^1(\Omega \setminus K^0)$, and let $u^1 \in L^2(\Omega)$. Assume that*

$$s^0 + c_0 T < L_2, \tag{3.1}$$

where c_0 is the constant in (2.6). Then there exists an admissible dynamic evolution $t \mapsto (\sigma(t), u(t))$ satisfying the initial condition $(\sigma(0), u(0), \dot{u}(0)) = (s^0, u^0, u^1)$ and the η-maximal dissipation condition introduced in Definition 2.5.

The proof of Theorem 3.1 is based on the following continuity result, which requires a preliminary definition.

Definition 3.2. Given $T_0 < T_1$, the set of pairs (σ, u) which satisfy conditions $(\Sigma 1)$, $(\Sigma 2)$, (U1), and (U2) on the interval $[T_0, T_1]$ is denoted by $\mathcal{A}_{reg}(T_0, T_1)$.

Theorem 3.3. *Let $0 \leq T_0 < T_1 \leq T$, let $K^0 := \gamma([-L_1, s^0])$ with $-L_1 < s^0 < L_2$, let $u^0 \in H^1(\Omega \setminus K^0)$, and let $u^1 \in L^2(\Omega)$. Let (σ_n, u_n) be a sequence in $\mathcal{A}_{reg}(T_0, T_1)$ with $(\sigma_n(T_0), u_n(T_0), \dot{u}_n(T_0)) = (s^0, u^0, u^1)$. Assume that*

$$\sigma_n(t) \to \sigma(t) \quad \text{for every } t \in [T_0, T_1]. \tag{3.2}$$

Then σ satisfies ($\Sigma 1$) and ($\Sigma 2$). Assume, in addition, that $\sigma(T_1) < L_2$, and let K be defined by (2.1)*. Then there exist a subsequence of u_n, not relabelled, and a solution u of the wave equation on the time-dependent cracking domains $\Omega \setminus K(t)$, $t \in [T_0, T_1]$, with homogeneous Neumann boundary conditions and initial conditions $u(T_0) = u^0$ and $\dot{u}(T_0) = u^1$, such that*

$$\begin{aligned} &u_n(t) \to u(t) \text{ strongly in } H^1(\Omega \setminus \Gamma), \\ &\dot{u}_n(t) \to \dot{u}(t) \text{ strongly in } L^2(\Omega), \end{aligned} \tag{3.3}$$

for every $t \in [T_0, T_1]$.

Remark 3.4. It was proven in [10, Theorem 4.1 and Example 4.3] that, if the hypotheses of Theorem 3.3 hold without a kink, for example, if $\sigma_n(T_1) \leq 0$ for all n large enough or $\sigma(T_0) > 0$, then (3.3) holds for every $t \in [T_0, T_1]$.

The proof of Theorem 3.3 in the general case will be given in Sections 6, 7, and 8.

Proof of Theorem 3.1. We consider a subdivision $0 = T_0 < T_1 < \cdots < T_k = T$ such that $T_j - T_{j-1} < \eta$ for $j = 1, \dots, k$, and we construct the solution recursively on the intervals of the subdivision. First of all we consider the maximum problem

$$\max \Big\{ \int_{T_0}^{T_1} \sigma(t)\, dt : (\sigma, u) \in \mathcal{A}_{reg}(T_0, T_1),\, \sigma(T_0) = s^0,\, u(T_0) = u^0,\, \dot{u}(T_0) = u^1 \Big\}. \tag{3.4}$$

To prove the existence of a maximum point let $(\sigma_n, u_n) \in \mathcal{A}_{reg}(T_0, T_1)$ be a maximizing sequence with $\sigma_n(T_0) = s^0$, $u_n(T_0) = u^0$, and $\dot{u}_n(T_0) = u^1$. By (Σ_1), (2.4), and (2.6) the functions σ_n are 1-Lipschitz. Therefore, by the Ascoli-Arzelà Theorem there exists a nondecreasing Lipschitz function $\sigma^* \colon [T_0, T_1] \to \mathbb{R}$ such that

$$\sigma_n \to \sigma^* \quad \text{uniformly in } [T_0, T_1]. \tag{3.5}$$

It is easy to prove that σ^* satisfies ($\Sigma 1$), ($\Sigma 2$), and $\sigma^*(T_0) = s^0$. By (2.6) and (3.1) we have $\sigma^*(T_1) \le s^0 + c_0 T_1 < L_2$. Let $K^*(t) := \gamma([-L_1, \sigma^*(t)])$ for every $t \in [T_0, T_1]$.

By Theorem 3.3 there exists a solution u^* of the wave equation on the time-dependent cracking domains $\Omega \setminus K^*(t)$, $t \in [T_0, T_1]$, with homogeneous Neumann bondary conditions and initial conditions $u^*(T_0) = u^0$ and $\dot u^*(T_0) = u^1$, such that (3.3) holds. Since (σ_n, u_n) satisfy (2.9), from (3.3) and (3.5) we deduce that (σ^*, u^*) satisfies (2.9), too. This implies that $(\sigma^*, u^*) \in \mathcal{A}_{reg}(T_0, T_1)$ and by (3.5), it is a solution of problem (3.4).

We consider now the interval $[T_1, T_2]$ and the maximum problem

$$\max \Big\{ \int_{T_1}^{T_2} \sigma(t)\, dt : (\sigma, u) \in \mathcal{A}_{reg}(T_1, T_2),\ \sigma(T_1) = \sigma^*(T_1),\ u(T_1) = u^*(T_1),\ \dot u(T_1) = \dot u^*(T_1) \Big\}.$$

The existence of a maximizer can be proved as in the previous step, recalling that the inequalities $\sigma^*(T_1) \le s^0 + c_0 T_1 < L_2$ and (2.6) give $\sigma^*(T_2) \le s^0 + c_0 T_2 < L_2$.

Arguing recursively we can thus construct a pair (σ^*, u^*) such that for $j = 1, \dots, k$ its restriction to $[T_{j-1}, T_j]$ belongs to $\mathcal{A}_{reg}(T_{j-1}, T_j)$ and maximizes

$$\int_{T_{j-1}}^{T_j} \sigma(t)\, dt$$

among all $(\sigma, u) \in \mathcal{A}_{reg}(T_{j-1}, T_j)$ such that $\sigma(T_{j-1}) = \sigma^*(T_{j-1})$, $u(T_{j-1}) = u^*(T_{j-1})$, $\dot u(T_{j-1}) = \dot u^*(T_{j-1})$. Let $K^*(t) := \gamma([-L_1, \sigma^*(t)])$ for every $t \in [0, T]$. Using [11, Definition 2.15 and Theorem 2.17] it is easy to see that u^* is a solution of the wave equation on the time-dependent cracking domains $\Omega \setminus K^*(t)$, $t \in [0, T]$, with homogeneous Neumann bondary conditions and initial conditions $u^*(T_0) = u^0$ and $\dot u^*(T_0) = u^1$, and that the pair (σ^*, u^*) satisfies the energy dissipation balance (2.9) for every $t \in [0, T]$. Therefore $(\sigma^*, u^*) \in \mathcal{A}$. By the definition of $\mathcal{A}_{reg}(T_{j-1}, T_j)$ we have $\mathrm{sing}(\sigma^*) \subset \{T_1, \dots, T_{k-1}\}$.

To prove that $(\sigma^*, u^*) \in \mathcal{A}$ satisfies the η-maximal dissipation condition we fix $0 \le \tau_0 < \tau_1 \le T$ and an admissible dynamic evolution $(\widehat\sigma, \widehat u) \in \mathcal{A}$ with $\mathrm{sing}(\widehat\sigma) \subset \mathrm{sing}(\sigma^*)$ such that

(a) $(\widehat\sigma(t), \widehat u(t)) = (\sigma^*(t), u^*(t))$ for $0 \le t \le \tau_0$
(b) $\widehat\sigma(t) \ge \sigma^*(t)$ for $\tau_0 < t \le \tau_1$, with strict inequality on a sequence converging to τ_0,
(c) $\widehat\sigma(\tau_1) > \sigma^*(\tau_1) + \eta$,

and show that this leads to a contradiction.

Let $j \in \{1, \dots, k\}$ be such that $T_{j-1} \le \tau_0 < T_j$. Then (a) and (c), using the monotonicity of σ^*, imply that $\widehat\sigma(\tau_1) > \sigma^*(\tau_1) + \eta \ge \sigma^*(\tau_0) + \eta = \widehat\sigma(\tau_0) + \eta$. On the other hand, by (2.6) we have $\widehat\sigma(\tau_1) - \widehat\sigma(\tau_0) \le \tau_1 - \tau_0$, hence $\eta < \tau_1 - \tau_0$, which implies $\tau_1 > T_j$. Since $\mathrm{sing}(\widehat\sigma) \subset \mathrm{sing}(\sigma^*)$, the restriction of $(\widehat\sigma, \widehat u)$ to $[T_{j-1}, T_j]$ belongs to $\mathcal{A}_{reg}([T_{j-1}, T_j])$. Moreover, by (a) we have $\widehat\sigma(T_{j-1}) = \sigma^*(T_{j-1})$, $\widehat u(T_{j-1}) = u^*(T_{j-1})$, $\dot{\widehat u}(T_{j-1}) = \dot u^*(T_{j-1})$. Therefore by the maximality of (σ^*, u^*) we have that

$$\int_{T_{j-1}}^{T_j} \widehat\sigma(t) dt \le \int_{T_{j-1}}^{T_j} \sigma^*(t) dt\,.$$

On the other hand (b) implies that the opposite strict inequality holds. This contradiction concludes the proof of the η-maximality of (σ^*, u^*). □

4. Some perturbation results

In the proof of Theorem 3.3 we shall repeatedly use the following result, which deals with solutions of the wave equation on the time-independent domain $\Omega \setminus \Gamma$, with homogeneous Neumann boundary conditions on $\partial\Omega \setminus \Gamma$ and no condition on Γ. It shows that, if we perturb the initial times and the initial conditions, then the solutions satisfying an energy inequality remain close to the initial data of the unperturbed problem in a short time interval, whose length can be uniformly estimated from below.

For every open set $A \subset \mathbb{R}^2$, $v \in H^1(A \setminus \Gamma)$, and $w \in L^2(A)$ we define

$$\mathcal{E}(v, w, A) := \|\nabla v\|^2_{L^2(A \setminus \Gamma; \mathbb{R}^2)} + \|w\|^2_{L^2(A)}\,. \tag{4.1}$$

For every $a, b \in \mathbb{R}$ we set $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

Lemma 4.1. *Let $T_0 < T_1$, $t_0 \in [T_0, T_1]$, $u^0 \in H^1(\Omega \setminus \Gamma)$, and $u^1 \in L^2(\Omega)$. For every $\varepsilon > 0$ there exists $\delta > 0$ with the following property: if*

(a) $\tau_0 \in [T_0, T_1]$, $|\tau_0 - t_0| < \delta$, $\tau_\delta := (\tau_0 + \delta) \wedge T_1$,
(b) $v^0 \in H^1(\Omega \setminus \Gamma)$, $v^1 \in L^2(\Omega)$, $\mathcal{E}(v^0 - u^0, v^1 - u^1, \Omega) < \delta$,
(c) *v is a solution of the wave equation on $[\tau_0, \tau_\delta] \times (\Omega \setminus \Gamma)$ with homogeneous Neumann boundary conditions on $\partial\Omega \setminus \Gamma$ and initial conditions $v(\tau_0) = v^0$ and $\dot v(\tau_0) = v^1$,*
(d) *v satisfies the energy estimate*

$$\mathcal{E}(v(t), \dot v(t), \Omega) \leq \mathcal{E}(v^0, v^1, \Omega) \quad \textit{for every } t \in [\tau_0, \tau_\delta], \tag{4.2}$$

then

$$\mathcal{E}(v(t) - u^0, \dot v(t) - u^1, \Omega) < \varepsilon \tag{4.3}$$

for every $t \in [\tau_0, \tau_\delta]$.

Proof. For every $r > 0$ we fix an open set $U_r \subset \mathbb{R}^2$ with Lipschitz boundary such that

$$\{x \in \mathbb{R}^2 : d(x, \Gamma) < r\} \subset U_r \subset \{x \in \mathbb{R}^2 : d(x, \Gamma) < 2r\} \tag{4.4}$$

and we set

$$V_r := \Omega \cap U_r. \tag{4.5}$$

Fix $\varepsilon > 0$. Then by the absolute continuity of the integral there exists $r > 0$ such that

$$\mathcal{E}(u^0, u^1, V_{7r}) < \varepsilon. \tag{4.6}$$

Let u_r be the solution of the wave equation on the set $\mathbb{R} \times (\Omega \setminus \overline{V}_r)$ with Cauchy conditions $u_r(t_0) = u^0$, $\dot u_r(t_0) = u^1$, Dirichlet boundary condition $u_r(t) = u^0$ on $\Omega \cap \partial V_r$, and homogeneous Neumann condition on the rest of the boundary of $\Omega \setminus \overline{V}_r$.

Let $u_{r,7r}$ be the solution of the wave equation on the set $\mathbb{R} \times (V_{7r} \setminus \overline{V}_r)$ with Cauchy conditions $u_{r,7r}(t_0) = u^0$, $\dot u_{r,7r}(t_0) = u^1$, Dirichlet boundary condition $u_{r,7r}(t) = u^0$ on $\Omega \cap (\partial V_r \cup \partial V_{7r})$, and homogeneous Neumann condition on the rest of the boundary of $V_{7r} \setminus \overline{V}_r$. By the energy conservation on time-independent domains we have

$$\mathcal{E}(u_{r,7r}(t), \dot u_{r,7r}(t), V_{7r} \setminus \overline{V}_r) = \mathcal{E}(u^0, u^1, V_{7r} \setminus \overline{V}_r) \quad \text{for every } t \in \mathbb{R}. \tag{4.7}$$

By (4.4) and (4.5) for every $x \in V_{3r}$ we have $d(x, \partial V_{7r} \cap \Omega) > r$. By the finite speed of propagation, see [9, Theorem A.1] applied with $U = V_{7r} \setminus \overline{V}_r$, $S_1 = \Omega \cap (\partial V_r \cup \partial V_{7r})$ and $S_0 = \Omega \cap \partial V_r$, we have

$$u_r(t) = u_{r,7r}(t) \quad \text{on } V_{3r} \setminus \overline{V}_r \quad \text{for every } t_0 \leq t \leq t_0 + r. \tag{4.8}$$

Therefore by (4.7) we obtain

$$\mathcal{E}(u_r(t), \dot u_r(t), V_{3r} \setminus \overline{V}_r) \leq \mathcal{E}(u^0, u^1, V_{7r} \setminus \overline{V}_r) \quad \text{for every } t_0 \leq t \leq t_0 + r. \tag{4.9}$$

Since $u_r \in C^0(\mathbb{R}; H^1(\Omega \setminus \overline{V}_r))$ and $\dot u_r \in C^0(\mathbb{R}; L^2(\Omega \setminus \overline{V}_r))$ (see, e.g., [17, Thm. 8.2]), there exists $0 < \widehat{\delta} < r$ such that

$$\mathcal{E}(u_r(t) - u^0, \dot u_r(t) - u^1, \Omega \setminus \overline{V}_r) < \varepsilon^2 \quad \text{for } t_0 - \widehat{\delta} \leq t \leq t_0 + \widehat{\delta}. \tag{4.10}$$

For every $\tau_0 \in [T_0, T_1]$, $v^0 \in H^1(\Omega \setminus \overline{V}_r)$, and $v^1 \in L^2(\Omega)$ let v_r (resp. $v_{r,7r}$) be the solution of the wave equation on the set $\mathbb{R} \times (\Omega \setminus \overline{V}_r)$ (resp. on the set $\mathbb{R} \times (V_{7r} \setminus \overline{V}_r)$) with Cauchy conditions $v_r(\tau_0) = v^0$, $\dot v_r(\tau_0) = v^1$ on $\Omega \setminus \overline{V}_r$ (resp. $v_{r,7r}(\tau_0) = v^0$, $\dot v_{r,7r}(\tau_0) = v^1$ on $V_{7r} \setminus \overline{V}_r$), Dirichlet boundary condition $v_r(t) = v^0$ on $\Omega \cap \partial V_r$ and homogeneous Neumann condition on the rest of the boundary of $\Omega \setminus \overline{V}_r$ (resp. Dirichlet boundary condition $v_{r,7r}(t) = v^0$ on $\Omega \cap (\partial V_r \cup \partial V_{7r})$ and homogeneous Neumann condition on the rest of the boundary of $V_{7r} \setminus \overline{V}_r$). By the energy conservation on time-independent domains we have

$$\begin{aligned} &\mathcal{E}(v_r(t), \dot v_r, \Omega \setminus \overline{V}_r) = \mathcal{E}(v^0, v^1, \Omega \setminus \overline{V}_r), \\ &\mathcal{E}(v_{r,7r}(t), \dot v_{r,7r}(t), V_{7r} \setminus \overline{V}_r) = \mathcal{E}(v^0, v^1, V_{7r} \setminus \overline{V}_r), \end{aligned} \tag{4.11}$$

for every $t \in \mathbb{R}$.

By the continuous dependence on the Cauchy data of the solution of the wave equation on a time-independent domain (see Lemma 4.2 below) there exists $0 < \delta < (\frac{1}{2}\hat\delta) \wedge \varepsilon$ such that, if $\tau_0 \in [T_0, T_1]$, $v^0 \in H^1(\Omega \setminus \overline{V}_r)$, $v^1 \in L^2(\Omega)$,

$$|\tau_0 - t_0| < \delta, \quad \text{and} \quad \mathcal{E}(v^0 - u^0, v^1 - u^1, \Omega) < \delta < \varepsilon, \tag{4.12}$$

then

$$\mathcal{E}(v_r(t) - u_r(t), \dot v_r(t) - \dot u_r(t), \Omega \setminus \overline{V}_r) < \varepsilon \quad \text{for every } T_0 \leq t \leq T_1. \tag{4.13}$$

Let us fix now v as in the statement of the lemma. Since $\Gamma \subset V_r$ we have that v is a solution of the wave equation on the set $[\tau_0, \tau_\delta] \times (\Omega \setminus \overline{V}_r)$ with initial conditions $v(\tau_0) = v^0$, $\dot v(\tau_0) = v^1$, and homogeneous Neumann boundary condition on $\partial\Omega \setminus \overline{V}_r$.

Since by (4.4) and (4.5) for every $x \in \Omega \setminus V_{3r}$ we have $d(x, V_r) > r > \delta$, by the finite speed of propagation, see [9, Theorem A.1] applied with $U = \Omega \setminus \overline{V}_r$, $S_1 = \Omega \cap \partial V_r$ and $S_0 = \emptyset$, we obtain

$$v(t) = v_r(t) \quad \text{on } \Omega \setminus \overline{V}_{3r} \quad \text{for every } \tau_0 \le t \le \tau_\delta \,. \tag{4.14}$$

On the other hand, as in the proof of (4.8) we can show that

$$v_r(t) = v_{r,7r}(t) \quad \text{on } V_{3r} \setminus \overline{V}_r \quad \text{for every } \tau_0 \le t \le \tau_\delta \,. \tag{4.15}$$

By (4.2) and (4.14) we have

$$\mathcal{E}(v_r(t), \dot v_r(t), \Omega \setminus \overline{V}_{3r}) + \mathcal{E}(v(t), \dot v(t), V_{3r}) \le \mathcal{E}(v^0, v^1, \Omega) \quad \text{for every } \tau_0 \le t \le \tau_\delta. \tag{4.16}$$

Assume that (4.12) holds. Then, using (4.11), (4.15), and (4.16), for every $\tau_0 \le t \le \tau_\delta$ we obtain

$$\begin{aligned} &\mathcal{E}(v(t), \dot v(t), V_{3r}) \le \mathcal{E}(v^0, v^1, \Omega) - \mathcal{E}(v_r(t), \dot v_r(t), \Omega \setminus \overline{V}_{3r}) \\ &\le \mathcal{E}(v^0, v^1, \Omega) - \mathcal{E}(v_r(t), \dot v_r(t), \Omega \setminus \overline{V}_r) + \mathcal{E}(v_r(t), \dot v_r(t), \overline{V}_{3r} \setminus \overline{V}_r) \\ &\le \mathcal{E}(v^0, v^1, \Omega) - \mathcal{E}(v^0, v^1, \Omega \setminus \overline{V}_r) + \mathcal{E}(v^0, v^1, V_{7r} \setminus \overline{V}_r) = \mathcal{E}(v^0, v^1, V_{7r}) < 4\varepsilon \,, \end{aligned} \tag{4.17}$$

where the last inequality follows from (4.6) and (4.12) and from the triangle inequality for $\sqrt{\mathcal{E}}$.

Then by (4.14) for every $\tau_0 \le t \le \tau_\delta$

$$\begin{aligned} &\mathcal{E}(v(t) - u^0, \dot v(t) - u^1, \Omega) \\ &= \mathcal{E}(v(t) - u^0, \dot v(t) - u^1, V_{3r}) + \mathcal{E}(v_r(t) - u^0, \dot v_r(t) - u^1, \Omega \setminus \overline{V}_{3r}). \end{aligned} \tag{4.18}$$

By the triangle inequality for $\sqrt{\mathcal{E}}$, using (4.6) and (4.17) we obtain

$$\mathcal{E}(v(t) - u^0, \dot v(t) - u^1, V_{3r}) \le 9\varepsilon \quad \text{for every } \tau_0 \le t \le \tau_\delta \,. \tag{4.19}$$

Moreover, by (4.10) and (4.13) we have

$$\mathcal{E}(v_r(t) - u^0, \dot v_r(t) - u^1, \Omega \setminus \overline{V}_{3r}) \le 4\varepsilon \tag{4.20}$$

for every $t_0 - \widehat\delta \le t \le t_0 + \widehat\delta$. Since $\delta < \frac12 \widehat\delta$ and $|\tau_0 - t_0| \le \delta$, we deduce that (4.20) holds for every $\tau_0 \le t \le \tau_\delta$. By (4.18), (4.19), and (4.20) we finally obtain that

$$\mathcal{E}(v(t) - u^0, \dot v(t) - u^1, \Omega) \le 13\varepsilon \quad \text{for every } \tau_0 \le t \le \tau_\delta \,,$$

which concludes the proof. □

In the proof of Lemma 4.1 we have used the following version of the continuous dependence of the solutions of the wave equation on time-independent domains, which takes into account also the case of Cauchy conditions prescribed at different times.

Lemma 4.2. *Let U be a bounded open set in $\mathbb{R}^2$ and let $\partial_D U$ be a relatively open subset of the Lipschitz part of the boundary. Given $T_0 < T_1$, $t_0 \in [T_0, T_1]$, $u^0 \in H^1(U)$, $u^1 \in L^2(U)$, let u be the solution of the wave equation in $[T_0, T_1] \times U$ with Cauchy conditions $u(t_0) = u^0$ and $\dot u(t_0) = u^1$, Dirichlet boundary condition $u(t) = u^0$ on $\partial_D U$, and homogeneous Neumann boundary condition on the remaining part of ∂U. For every $\varepsilon > 0$ there exists $\delta > 0$ with the following property: if*

$$\tau_0 \in [T_0, T_1] \,, \quad |\tau_0 - t_0| < \delta \,, \tag{4.21}$$

$$v^0 \in H^1(U) \,, \quad v^1 \in L^2(U) \,, \quad \mathcal{E}(v^0 - u^0, v^1 - u^1, U) < \delta \,, \tag{4.22}$$

and v is the solution of the wave equation on $[T_0, T_1] \times U$, with Cauchy conditions $v(\tau_0) = v^0$ and $\dot v(\tau_0) = v^1$, Dirichlet condition $v = v^0$ on $\partial_D U$, and homogeneous Neumann boundary condition on the remaining part of ∂U, then

$$\mathcal{E}(v(t) - u(t), \dot v(t) - \dot u(t), U) < \varepsilon \tag{4.23}$$

for every $t \in [T_0, T_1]$.

Proof. By existence and uniqueness of the solution to the Cauchy problem for the wave equation in time-independent domains, we can extend u and v to $[T_0-1,T_1+1]\times U$ in such a way that the extensions, still denoted by u and v, satisfy the wave equation on $[T_0-1,T_1+1]\times U$ with the same boundary conditions on $\partial_D U$ and $\partial U\setminus\partial_D U$. Assuming that $|\tau_0-t_0|<\frac12$ the function

$$v_*(t):=v(t-t_0+\tau_0)$$

is well defined for every $t\in[T_0-\frac12,T_1+\frac12]$ and satisfies the wave equation on $[T_0-\frac12,T_1+\frac12]\times U$ with Cauchy condition $v_*(t_0)=v^0$, $\dot v_*(t_0)=v^1$, Dirichlet boundary condition $v_*(t)=v^0$ on $\partial_D U$ and homogeneous Neumann boundary condition.

By the continuous dependence of the solutions on the Cauchy data (see, e.g., [17, Thm. 8.2]), there exists $\delta>0$ such that, if (4.22) is satisfied, then

$$\mathcal{E}(v_*(t)-u(t),\dot v_*(t)-\dot u(t),U)<\tfrac14\varepsilon\quad\text{for every } t\in[T_0-\tfrac12,T_1+\tfrac12]\,. \tag{4.24}$$

To obtain (4.23) from the previous inequality we introduce the function u_* defined by $u_*(t)=u(t-t_0+\tau_0)$ for every $t\in[T_0-\frac12,T_1+\frac12]$. Since u is the solution of the wave equation on a time-independent domain with time-independent boundary conditions we have that $u\in C^0([T_0-1,T_1+1];H^1(U))$ and $\dot u\in C^0([T_0-1,T_1+1];L^2(U))$ (see, e.g., [17, Thm. 8.2]). Therefore, by uniform continuity (and possibly reducing δ) we may assume that if (4.21) is satisfied then

$$\mathcal{E}(u_*(t)-u(t),\dot u_*(t)-\dot u(t),U)<\tfrac14\varepsilon,\quad\text{for every } t\in[T_0-\tfrac12,T_1+\tfrac12]\,. \tag{4.25}$$

By the triangle inequality for $\sqrt{\mathcal{E}}$ we obtain

$$\mathcal{E}(v_*(t)-u_*(t),\dot v_*(t)-\dot u_*(t),U)<\varepsilon\quad\text{for every } t\in[T_0-\tfrac12,T_1+\tfrac12]\,. \tag{4.26}$$

It is not restrictive to assume also $\delta<\frac12$, therefore (4.26) implies that (4.23) holds for every $t\in[T_0,T_1]$. ∎

5. A compactness result

Given $T_0<T_1$, consider the set $\mathcal{A}_{reg}(T_0,T_1)$ introduced in Definition 3.2 and a sequence

$$(\sigma_n,u_n)\in\mathcal{A}_{reg}(T_0,T_1)\,. \tag{5.1}$$

Assume that there exist $s^0\in(-L_1,L_2)$, $u^0\in H^1(\Omega\backslash K^0)$ with $K^0:=\gamma([-L_1,s^0])$, and $u^1\in L^2(\Omega)$ such that

$$\sigma_n(T_0)=s^0\,,\quad u_n(T_0)=u^0\,,\quad \dot u_n(T_0)=u^1 \tag{5.2}$$

for every n. Assume that

$$\sigma_n(t)\to\sigma(t)\quad\text{for every } t\in[T_0,T_1]\,, \tag{5.3}$$

and let $K(t)$ be defined by

$$K(t):=\gamma([-L_1,\sigma(t)])\quad\text{for every } t\in[T_0,T_1]\,. \tag{5.4}$$

The main result of this section is the following theorem.

Theorem 5.1. *Assume* (5.1)-(5.4). *Then there exist a subsequence, not relabelled, and a solution u of the wave equation on $\Omega\setminus K(t)$, $T_0\le t\le T_1$, with homogeneous Neumann boundary condition, such that*

$$\begin{aligned} &u_n\rightharpoonup u\quad\textit{weakly in } L^2((T_0,T_1);H^1(\Omega\setminus\Gamma))\,,\\ &\dot u_n\rightharpoonup \dot u\quad\textit{weakly in } L^2((T_0,T_1);L^2(\Omega))\,. \end{aligned} \tag{5.5}$$

Moreover, after a modification on a set of measure zero,

$$u\in C_w([T_0,T_1];H^1(\Omega\setminus\Gamma))\cap C([T_0,T_1];L^2(\Omega))\quad\textit{and}\quad \dot u\in C_w([T_0,T_1];L^2(\Omega))\,. \tag{5.6}$$

Finally

$$u(T_0)=u^0\quad\textit{and}\quad \dot u(T_0)=u^1\,. \tag{5.7}$$

To prove this result we shall use the following lemmas.

Lemma 5.2. *Let $u,v\in H^1((T_0,T_1);L^2(\Omega))$ and set $z:=u\vee v$. Then $z\in H^1((T_0,T_1);L^2(\Omega))$ and for a.e. $t\in(T_0,T_1)$ we have*

$$\dot z(t)=\begin{cases}\dot u(t) & \textit{a.e. on } \{x\in\Omega: u(t,x)>v(t,x)\}\,,\\ \dot u(t)=\dot v(t) & \textit{a.e. on } \{x\in\Omega: u(t,x)=v(t,x)\}\,,\\ \dot v(t) & \textit{a.e. on } \{x\in\Omega: u(t,x)\le v(t,x)\}\,.\end{cases} \tag{5.8}$$

Proof. It is enough to prove the result when $v=0$, so that $z=u\vee 0$. Since $s\mapsto s\vee 0$ is 1-Lipschitz from $\mathbb{R}$ to $\mathbb{R}$, we have $\|z(t_1)-z(t_2)\|_{L^2(\Omega)}\le\|u(t_1)-u(t_2)\|_{L^2(\Omega)}$ for every $t_1,t_2\in[T_0,T_1]$. Since, after a modification on a set of measure zero, $u\in AC([T_0,T_1];L^2(\Omega))$, the previous inequality implies that $z\in AC([T_0,T_1];L^2(\Omega))$ and that $\|\dot z(t)\|_{L^2(\Omega)}\le\|\dot u(t)\|_{L^2(\Omega)}$ for a.e. $t\in(T_0,T_1)$. Hence $\dot z\in L^2((T_0,T_1);L^2(\Omega))$, which gives $z\in H^1((T_0,T_1);L^2(\Omega))$.

Let us prove (5.8) at a time $t\in(T_0,T_1)$ such that $\dot u(t)$ and $\dot z(t)$ are the strong limit in L^2 of the corresponding difference quotients. Then there exists a sequence $h_j\to 0+$ such that

$$\frac{u(t+h_j,x)-u(t,x)}{h_j}\to\dot u(t,x)\quad\text{and}\quad\frac{z(t+h_j,x)-z(t,x)}{h_j}\to\dot z(t,x) \tag{5.9}$$

for a.e. $x\in\Omega$. Similarly, there exists a sequence $h_j\to 0-$ such that (5.9) still holds for a.e. $x\in\Omega$. This implies that $\dot z(t)=\dot u(t)$ a.e. on $\{x\in\Omega:u(t,x)>0\}$ and $\dot z(t)=0$ a.e. on $\{x\in\Omega:u(t,x)<0\}$.

Let us fix now a point x in the set $\{x\in\Omega:u(t,x)=0\}$ such that (5.9) holds for a sequence $h_j\to 0+$ and also for a sequence $h_j\to 0-$. Since $z(t,x)=0$ and $z(t+h_j,x)\ge 0$ from the second formula in (5.9) we obtain that $\dot z(t,x)\ge 0$ when $h_j\to 0+$ and $\dot z(t,x)\le 0$ when $h_j\to 0-$, hence $\dot z(t,x)=0$. Since $u(t,x)=z(t,x)=0$, we have $u(t+h_j,x)-u(t,x)\le z(t+h_j,x)-z(t,x)$. By (5.9), this implies that $\dot z(t,x)\ge\dot u(t,x)$ when $h_j\to 0+$ and $\dot z(t,x)\le\dot u(t,x)$ when $h_j\to 0-$, hence $\dot z(t,x)=\dot u(t,x)$. This concludes the proof of (5.8). □

We now prove an approximation result that will be used in the sequel.

Lemma 5.3. *Assume* (5.1)-(5.4) *and let* $\varphi\in C^0([T_0,T_1];H^1(\Omega\setminus\Gamma))\cap C^1([T_0,T_1];L^2(\Omega))$ *with* $\varphi(t)\in H^1(\Omega\setminus K(t))$ *for every* $t\in[T_0,T_1]$. *Assume that there exists* $M>0$ *such that* $\|\varphi(t)\|_{L^\infty(\Omega)}\le M$ *for every* $t\in[T_0,T_1]$. *Then for every* $\varepsilon>0$ *there exist* $\varphi^\varepsilon\in L^2((T_0,T_1);H^1(\Omega\setminus\Gamma))\cap H^1((T_0,T_1);L^2(\Omega))$ *and* $n_\varepsilon>0$ *such that*

$$\|\varphi^\varepsilon-\varphi\|_{L^2((T_0,T_1);H^1(\Omega\setminus\Gamma))}<\varepsilon\quad\textit{and}\quad\|\dot\varphi^\varepsilon-\dot\varphi\|_{L^2((T_0,T_1);L^2(\Omega))}<\varepsilon\,, \tag{5.10}$$

and $\varphi^\varepsilon(t)\in H^1(\Omega\setminus K_n(t))$ *for a.e.* $t\in(T_0,T_1)$ *and for every* $n\ge n_\varepsilon$.

Proof. For every $R>0$ and every $x\in\mathbb{R}^2$ the open ball with centre x and radius R is denoted by $B_R(x)$. For every $t\in[T_0,T_1]$ we set

$$U_R(t):=B_R(\gamma(\sigma(t)))\,. \tag{5.11}$$

Since $\varphi\in C^0([T_0,T_1];H^1(\Omega\setminus\Gamma))$ and $\dot\varphi\in C^0([T_0,T_1];L^2(\Omega))$, by the absolute continuity of the integral we may fix $R>0$ such that

$$\|\varphi(t)\|_{H^1(U_R(t))}<\varepsilon\quad\text{and}\quad\|\dot\varphi(t)\|_{L^2(U_R(t))}<\varepsilon\quad\text{for every }t\in[T_0,T_1]\,. \tag{5.12}$$

We may also assume that

$$|U_R(t)|<\varepsilon^2. \tag{5.13}$$

Using the capacitary potential of the ball $B_r(0)$ with respect to the ball $B_R(0)$ we can find a radius $0<r<R$ and a function $\psi_0^\varepsilon\in C^\infty(\mathbb{R}^2)$ such that

$$\begin{aligned}&\psi_0^\varepsilon(x)=0\quad\text{if }|x|\le r\,,\\&0\le\psi_0^\varepsilon(x)\le M\,,\\&\psi_0^\varepsilon(x)=M\quad\text{if }|x|\ge R\,,\\&\|\psi_0^\varepsilon\|_{H^1(B_R(0))}<\varepsilon\,.\end{aligned} \tag{5.14}$$

We define $\psi^\varepsilon(t,x):=\psi_0^\varepsilon(x-\gamma(\sigma(t)))$. Since $\dot\psi^\varepsilon(t,x)=-\nabla\psi_0^\varepsilon(x-\gamma(\sigma(t)))\dot\gamma(\sigma(t))\dot\sigma(t)$, we have

$$\|\dot\psi^\varepsilon(t)\|_{L^2(\Omega)}\le\|\nabla\psi_0^\varepsilon\|_{L^2(\Omega;\mathbb{R}^2)}<\varepsilon\,. \tag{5.15}$$

Notice that $\psi^\varepsilon\in L^2((T_0,T_1);H^1(\Omega\setminus\Gamma))\cap H^1((T_0,T_1);L^2(\Omega))$. Moreover, $\psi^\varepsilon(t,x)=0$ for $x\in U_r(t)$ and $\psi^\varepsilon(t,x)=M$ for $x\in\Omega\setminus U_R(t)$.

We define $\varphi^\varepsilon(t,x):=(\varphi(t,x)\wedge\psi^\varepsilon(t,x))\vee(-\psi^\varepsilon(t,x))$ for every $t\in[T_0,T_1]$ and every $x\in\Omega$. By Lemma 5.2 we have $\varphi^\varepsilon\in L^2((T_0,T_1);H^1(\Omega\setminus\Gamma))\cap H^1((T_0,T_1);L^2(\Omega))$. Since $\psi^\varepsilon(t)\in H^1(\Omega)$ for every $t\in[T_0,T_1]$ we have $\varphi^\varepsilon(t)\in H^1(\Omega\setminus K(t))$. Moreover, $\varphi^\varepsilon(t,x)=0$ for $x\in U_r(t)$ and $\varphi^\varepsilon(t,x)=\varphi(t,x)$ for $x\in\Omega\setminus U_R(t)$. By (5.3) there exists n_ε such that $K_n(t)\Delta K(t)\subset U_r(t)$ for every $t\in[T_0,T_1]$ and for every $n\ge n_\varepsilon$. Therefore, the previous remarks give $\varphi^\varepsilon(t)\in H^1(\Omega\setminus K_n(t))$.

It remains to prove (5.10). Since $\varphi^\varepsilon(t) = \varphi(t)$ on $\Omega \setminus U_R(t)$, we have $\|\varphi^\varepsilon(t) - \varphi(t)\|_{H^1(\Omega\setminus\Gamma)} = \|\varphi^\varepsilon(t) - \varphi(t)\|_{H^1(U_R(t)\setminus\Gamma)} \le \|\varphi(t)\|_{H^1(U_R(t)\setminus\Gamma)} + \|\psi^\varepsilon(t)\|_{H^1(U_R(t)\setminus\Gamma)}$. Since by (5.14) we have

$$\int_{U_R(t)} |\nabla\psi^\varepsilon(t)|^2 < \varepsilon^2 \quad \text{and} \quad \int_{U_R(t)} |\psi^\varepsilon(t)|^2 \le M^2 |U_R(t)| \le M^2\varepsilon^2\,,$$

using (5.12) we obtain $\|\varphi^\varepsilon(t) - \varphi(t)\|_{H^1(\Omega\setminus\Gamma)} \le \varepsilon(2+M)$. Integrating over $[T_0, T_1]$ we obtain the first inequality in (5.10) with ε replaced by $\varepsilon(2+M)\sqrt{T_1 - T_0}$.

As for the time-derivatives, by Lemma 5.2 we have $\|\dot\varphi^\varepsilon(t) - \dot\varphi(t)\|_{L^2(\Omega)} = \|\dot\varphi^\varepsilon(t) - \dot\varphi(t)\|_{L^2(U_R(t))} \le \|\dot\varphi(t)\|_{L^2(U_R(t))} + \|\dot\psi^\varepsilon(t)\|_{L^2(U_R(t))}$ and by (5.12) and (5.15), we have $\|\dot\varphi^\varepsilon(t) - \dot\varphi(t)\|_{L^2(\Omega)} \le 2\varepsilon$, hence integrating over $[T_0, T_1]$ we obtain the second inequality in (5.10) with ε replaced by $2\varepsilon\sqrt{T_1 - T_0}$. □

Proof of Theorem 5.1. By (2.9) the functions $\nabla u_n(t)$ and $\dot u_n(t)$ are bounded in $L^2(\Omega\setminus\Gamma; \mathbb{R}^2)$ and $L^2(\Omega)$ uniformly in t and n. Integrating $\dot u_n$ with respect to time and taking the initial conditions (5.2) into account, we obtain also that the functions $u_n(t)$ are bounded in $L^2(\Omega)$ uniformly in t and n. This implies that

$$\begin{aligned} &u_n \quad \text{is bounded in } L^\infty((T_0, T_1); H^1(\Omega\setminus\Gamma))\,, \\ &\dot u_n \quad \text{is bounded in } L^\infty((T_0, T_1); L^2(\Omega))\,, \end{aligned} \tag{5.16}$$

uniformly in n. Therefore there exists

$$u \in L^\infty((T_0, T_1); H^1(\Omega\setminus\Gamma)) \quad \text{with} \quad \dot u \in L^\infty((T_0, T_1); L^2(\Omega)) \tag{5.17}$$

such that a subsequence (not relabelled) satisfies

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{weakly in } L^2((T_0, T_1); H^1(\Omega\setminus\Gamma))\,, \\ &\dot u_n \rightharpoonup \dot u \quad \text{weakly in } L^2((T_0, T_1); L^2(\Omega))\,. \end{aligned} \tag{5.18}$$

By (5.2) we have

$$u_n(t) = u^0 + \int_{T_0}^t \dot u_n(\tau)d\tau\,.$$

Let us define

$$v(t) := u^0 + \int_{T_0}^t \dot u(\tau)d\tau\,.$$

By (5.18) we have $u_n(t) \rightharpoonup v(t)$ weakly in $L^2(\Omega)$ for every $t \in [T_0, T_1]$. By (5.16) this implies that $u_n \rightharpoonup v$ weakly in $L^2((T_0, T_1); L^2(\Omega))$. Therefore (5.18) gives $u(t) = v(t)$ for a.e. $t \in (T_0, T_1)$.

Hence, after a modification on a set of measure zero, we have

$$u \in C([T_0, T_1]; L^2(\Omega)) \tag{5.19}$$

and the first equality in (5.7) holds.

Let us prove that

$$u(t) \in H^1(\Omega\setminus K(t)) \quad \text{for a.e. } t \in (T_0, T_1)\,. \tag{5.20}$$

For every $\varepsilon > 0$ let

$$K^\varepsilon(t) := \gamma([-L_1, (\sigma(t) + \varepsilon) \wedge L_2])\,. \tag{5.21}$$

By (5.3) there exists n_ε such that $K_n(t) \subset K^\varepsilon(t)$ for every $t \in [T_0, T_1]$ and $n \ge n_\varepsilon$, hence $u_n(t) \in H^1(\Omega\setminus K^\varepsilon(t))$ for a.e. $t \in (T_0, T_1)$ and for every $n \ge n_\varepsilon$. Since

$$\{v \in L^2((T_0, T_1); H^1(\Omega\setminus\Gamma)) : v(t) \in H^1(\Omega\setminus K^\varepsilon(t)) \quad \text{for a.e. } t \in (T_0, T_1)\} \tag{5.22}$$

is a linear subspace of $L^2((T_0, T_1); H^1(\Omega\setminus\Gamma))$ which is closed in the strong topology, it is closed also in the weak topology. This implies that $u(t) \in H^1(\Omega\setminus K^\varepsilon(t))$ for a.e. $t \in (T_0, T_1)$. Since $\bigcap_\varepsilon K^\varepsilon(t) = K(t)$ by (5.4), we obtain that $u(t) \in H^1(\Omega\setminus K(t))$ for a.e. $t \in (T_0, T_1)$.

We claim that u is a solution of the wave equation on $\Omega\setminus K(t)$, $T_0 \le t \le T_1$ with homogeneous Neumann boundary condition. By [11, Remark 2.9] we have to show that

$$\int_{T_0}^{T_1} (\dot u(t), \dot\varphi(t))dt = \int_{T_0}^{T_1} (\nabla u(t), \nabla\varphi(t))dt \tag{5.23}$$

for every $\varphi \in C_c^\infty((T_0,T_1);H^1(\Omega\setminus\Gamma))\cap H^1((T_0,T_1);L^2(\Omega))$ and with $\varphi(t)\in H^1(\Omega\setminus K(t))$ for every $t\in[T_0,T_1]$. Given such a φ, for every $M>0$ we define

$$\varphi^M(t) := (\varphi(t)\wedge M)\vee(-M)\,.$$

Using Lemma 5.2 it is easy to show that φ^M satisfies the hypotheses of Lemma 5.3. Therefore for every $\varepsilon>0$ there exist $\varphi^{M,\varepsilon}\in L^2((T_0,T_1);H^1(\Omega\setminus\Gamma))\cap H^1((T_0,T_1);L^2(\Omega))$ and $n_\varepsilon>0$ such that (5.10) holds and $\varphi^{M,\varepsilon}\in H^1(\Omega\setminus K_n(t))$ for a.e. $t\in(T_0,T_1)$ and for every $n\geq n_\varepsilon$.

It is not restrictive to assume that $\varphi^{M,\varepsilon}(T_0)=\varphi^{M,\varepsilon}(T_1)=0$. Indeed, if it is not so we can replace $\varphi^{M,\varepsilon}(t)$ by $\omega(t)\varphi^{M,\varepsilon}(t)$, where $\omega\in C_c^\infty(T_0,T_1)$ and $\omega(t)=1$ for $t\in\operatorname{supp}(\varphi)$. Since u_n satisfies the wave equation on $\Omega\setminus K_n(t)$, $T_0\leq t\leq T_1$, with homogeneous Neumann boundary conditions, we have

$$\int_{T_0}^{T_1}(\dot u_n(t),\dot\varphi^{M,\varepsilon}(t))dt=\int_{T_0}^{T_1}(\nabla u_n(t),\nabla\varphi^{M,\varepsilon}(t))dt\,.$$

Passing to the limit as $n\to\infty$ we obtain

$$\int_{T_0}^{T_1}(\dot u(t),\dot\varphi^{M,\varepsilon}(t))dt=\int_{T_0}^{T_1}(\nabla u(t),\nabla\varphi^{M,\varepsilon}(t))dt \tag{5.24}$$

Passing to the limit as $\varepsilon\to 0$ using (5.10) and then as $M\to+\infty$ we obtain (5.23).

Therefore, by (5.17) we can use [11, Proposition 2.18] and we obtain that, after a modification on a set of measure zero,

$$u\in C_w([T_0,T_1];H^1(\Omega\setminus\Gamma))\quad\text{and}\quad \dot u\in C_w([T_0,T_1];L^2(\Omega))\,,$$

which together with (5.19) concludes the proof of (5.6).

It remains to prove the second equality in (5.7). We apply Lemma 4.1 with $t_0:=T_0$, $\tau_0:=T_0$, $v^0:=u^0$, $v^1:=u^1$, and $v(t):=u_n(t)$. Note that u_n satisfies (4.2) by (2.9), which follows from (5.1) and from Definition 3.2. Therefore for every $\varepsilon>0$ there exists $0<\delta<T_1-T_0$ such that

$$\mathcal{E}(u_n(t)-u^0,\dot u_n(t)-u^1,\Omega)<\varepsilon \tag{5.25}$$

for every $t\in[T_0,T_0+\delta]$. For every interval $[a,b]\subset[T_0,T_0+\delta]$ we obtain

$$\int_a^b\mathcal{E}(u_n(t)-u^0,\dot u_n(t)-u^1,\Omega)dt<\varepsilon(b-a)\,.$$

Since $u_n\rightharpoonup u$ weakly in $L^2((T_0,T_1);H^1(\Omega\setminus\Gamma))$ and $\dot u_n\rightharpoonup\dot u$ weakly in $L^2((T_0,T_1);L^2(\Omega))$, by lower semicontinuity we have

$$\int_a^b\mathcal{E}(u(t)-u^0,\dot u(t)-u^1,\Omega)dt\leq\varepsilon(b-a)\,,$$

hence

$$\mathcal{E}(u(t)-u^0,\dot u(t)-u^1,\Omega)\leq\varepsilon$$

for a.e. $t\in[T_0,T_0+\delta]$. By (5.6) this inequality holds for every $t\in[T_0,T_0+\delta]$. Since ε is arbitrary this proves the second equality in (5.7). ☐

6. Strong convergence approaching the kink

Assume that (σ_n,u_n), σ, and K satisfy conditions (5.1)-(5.4). In view of Theorem 5.1 we also assume that there exists a solution u of the wave equation on $\Omega\setminus K(t)$, $T_0\leq t\leq T_1$, with Neumann boundary condition and with

$$u\in C_w([T_0,T_1];H^1(\Omega\setminus\Gamma))\cap C([T_0,T_1];L^2(\Omega))\quad\text{and}\quad\dot u\in C_w([T_0,T_1];L^2(\Omega))\,, \tag{6.1}$$

$$u(T_0)=u^0\quad\text{and}\quad\dot u(T_0)=u^1 \tag{6.2}$$

such that

$$\begin{aligned} &u_n\rightharpoonup u\quad\text{weakly in } L^2((T_0,T_1);H^1(\Omega\setminus\Gamma))\,,\\ &\dot u_n\rightharpoonup\dot u\quad\text{weakly in } L^2((T_0,T_1);L^2(\Omega))\,. \end{aligned} \tag{6.3}$$

According to (2.2) we define

$$\tau_1=\sup\{t\in[T_0,T_1]:\sigma(t)<0\}\,, \tag{6.4}$$

with the convention $\sup\emptyset=T_0$.

This section deals with the strong convergence of $u_n(t)$ and $\dot u_n(t)$ for $T_0 \le t \le \tau_1$. The main result is the following theorem.

Theorem 6.1. *Assume* (5.1)-(5.4) *and* (6.1)-(6.4), *with* $T_0 < \tau_1$. *Then*

$$\begin{aligned} &u_n(t) \to u(t) \text{ strongly in } H^1(\Omega \setminus \Gamma)\,,\\ &\dot u_n(t) \to \dot u(t) \text{ strongly in } L^2(\Omega)\,, \end{aligned} \tag{6.5}$$

for every $t \in [T_0, \tau_1]$.

In the case $t \in [T_0, \tau_1)$ the strong convergence follows from the continuous dependence of the solutions on the cracks. To prove the result when $t = \tau_1$, which will be crucial in our analysis of the case $t > \tau_1$ in Sections 6 and 8, we first prove the following result, which will be useful also in the case $\tau_1 = T_0$.

According to (2.2) we define

$$\tau_1^n = \sup\{t \in [T_0, T_1] : \sigma^n(t) < 0\}\,, \tag{6.6}$$

with the convention $\sup \emptyset = T_0$. Up to a subsequence we may assume that

$$\tau_1^n \to \tau_1^\infty\,. \tag{6.7}$$

By (5.3) we have $\tau_1 \le \tau_1^\infty$.

Proposition 6.2. *Assume* (5.1)-(5.4), (6.1)-(6.4), *and* (6.6), *with* $T_0 < \tau_1$. *Let* $t_n \in [T_0, \tau_1^n]$ *and assume that* $t_n \to t_\infty \in [T_0, \tau_1]$. *Then*

$$\begin{aligned} &u_n(t_n) \to u(t_\infty) \text{ strongly in } H^1(\Omega \setminus \Gamma)\,,\\ &\dot u_n(t_n) \to \dot u(t_\infty) \text{ strongly in } L^2(\Omega)\,. \end{aligned} \tag{6.8}$$

Proof. In the case $t_\infty = T_0$ we apply Lemma 4.1 with $t_0 := \tau_0 := T_0$, $0 < \delta < T_1 - T_0$, $v^1 := u^1$, and $v(t) := u_n(t)$. The energy inequality (4.2) for v follows from the energy equality (2.9) for (K_n, u_n), which is a consequence of (5.1) and Definition 3.2.

Therefore, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mathcal{E}(u_n(t) - u^0, \dot u_n(t) - u^1, \Omega) \le \varepsilon$$

for every $t \in [T_0, T_0 + \delta]$. Since $t_n \to T_0$ we have

$$\mathcal{E}(u_n(t_n) - u^0, \dot u_n(t_n) - u^1, \Omega) \le \varepsilon$$

for n large enough. By the arbitrariness of ε and by (6.2) this implies (6.8) for $t_\infty = T_0$.

In the case $t_\infty > T_0$ the result follows from Proposition 6.3 below. ☐

Let $\lambda : [a, b] \to \Omega$ be a $C^{3,1}$ curve and set $\Lambda = \lambda([a, b])$. Let $T_0 \le t_0^n < t_1^n \le T_1$ and let $\sigma^n : [t_0^n, t_1^n] \to [a, b]$ be $C^{3,1}$ functions which satisfy (2.6) and (2.7) for some constants $M > 0$ and $c_0 \in (0, 1)$ independent of n. Assume

$$t_0^n \to t_0^\infty \qquad t_1^n \to t_1^\infty \qquad t_0^\infty < t_1^\infty \tag{6.9}$$

and

$$\sigma^n(t) \to \sigma^\infty(t) \quad \text{for every } t \in (t_0^\infty, t_1^\infty), \tag{6.10}$$

where $\sigma^\infty : [t_0^\infty, t_1^\infty] \to [a, b]$ is of class $C^{3,1}$ and satisfies (2.6) and (2.7) with the same constants M and δ.

For every $t \in [t_0^n, t_1^n]$ let $H_n(t) := \lambda([a, \sigma^n(t)])$, and for every $t \in [t_0^\infty, t_1^\infty]$ let $H_\infty(t) := \lambda([a, \sigma^\infty(t)])$.

Proposition 6.3. *For every* n *let* $v_n^0 \in H^1(\Omega \setminus H_n(t_0^n))$ *and let* $v_n^1 \in L^2(\Omega)$. *Assume that*

$$\begin{aligned} &v_n^0 \to v_\infty^0 \quad \text{strongly in } H^1(\Omega \setminus \Lambda)\,,\\ &v_n^1 \to v_\infty^1 \quad \text{strongly in } L^2(\Omega)\,, \end{aligned}$$

with $v_\infty^0 \in H^1(\Omega \setminus H(t_0^\infty))$. *Let* v_n *be the solution of the wave equation on* $\Omega \setminus H_n(t)$, $t_0^n \le t \le t_1^n$ *with Neumann boundary condition and initial conditions* $v_n(t_0^n) = v_n^0$ *and* $\dot v_n(t_0^n) = v_n^1$, *let* v_∞ *be* *the solution of the wave equation on* $\Omega \setminus H(t)$, $t_0^\infty \le t \le t_1^\infty$ *with Neumann boundary condition and initial conditions* $v_\infty(t_0^\infty) = v_\infty^0$ *and* $\dot v_\infty(t_0^\infty) = v_\infty^1$, *and let* t_n *be a sequence such that*

$$t_n \to t_\infty\,, \tag{6.11}$$

$$t_0^n \le t_n \le t_1^n \tag{6.12}$$

for some $t_\infty \in (t_0^\infty, t_1^\infty]$. *Then*

$$\begin{aligned} &v_n(t_n) \to v_\infty(t_\infty) \text{ strongly in } H^1(\Omega \setminus \Lambda)\,,\\ &\dot v_n(t_n) \to \dot v_\infty(t_\infty) \text{ strongly in } L^2(\Omega)\,. \end{aligned} \tag{6.13}$$

Proof. By a translation in time we reduce the problem to the case of a fixed initial time. Let $\varepsilon_n := t_0^n - t_0^\infty$ and $v_n^\sharp(t) := v_n(t+\varepsilon_n)$. Then $v_n^\sharp$ is the solution of the wave equation on $\Omega \setminus H_n(t+\varepsilon_n)$, for $t \in [t_0^n, t_1^n - \varepsilon_n]$ with Neumann boundary condition and initial conditions $v_n^\sharp(t_0^\infty) = v_n^0$ and $\dot v_n^\sharp(t_0^\infty) = v_n^1$. Let us fix $t_0 \in (t_0^\infty, t_\infty)$. By [10, Theorem 4.1 and Example 4.3] applied on $[t_0^\infty, t_0]$ we have

$$\begin{aligned} &v_n^\sharp(t) \to v(t) \text{ strongly in } H^1(\Omega \setminus \Lambda)\,,\\ &\dot v_n^\sharp(t) \to \dot v(t) \text{ strongly in } L^2(\Omega)\,. \end{aligned}$$

for every $t \in [t_0^\infty, t_0]$. By the bounds on the energies we obtain

$$\begin{aligned} &v_n^\sharp \to v \text{ strongly in } L^2([t_0^\infty, t_0]; H^1(\Omega \setminus \Lambda))\,,\\ &\dot v_n^\sharp \to \dot v \text{ strongly in } L^2([t_0^\infty, t_0]; L^2(\Omega))\,. \end{aligned} \tag{6.14}$$

Let $\eta_n := t_n - t_\infty$ and $v_n^\flat(t) := v_n(t+\eta_n) = v_n^\sharp(t+\eta_n-\varepsilon_n)$ for $t \in [t_0^n - \eta_n, t_1^n - \eta_n]$, so that $v_n^\flat(t_\infty) = v_n(t_n)$. Let us fix $\hat\tau_0$, $\hat t_0$ such that $t_0^\infty < \hat\tau_0 < \hat t_0 < t_\infty$. Then by (6.14) we have

$$\begin{aligned} &v_n^\flat \to v \text{ strongly in } L^2([\hat\tau_0, \hat t_0]; H^1(\Omega \setminus \Lambda))\,,\\ &\dot v_n^\flat \to \dot v \text{ strongly in } L^2([\hat\tau_0, \hat t_0]; L^2(\Omega))\,. \end{aligned}$$

Hence there exists $\tau \in [\hat\tau_0, \hat t_0]$ such that (for a subsequence, not relabelled)

$$\begin{aligned} &v_n^\flat(\tau) \to v(\tau) \text{ strongly in } H^1(\Omega \setminus \Lambda)\,,\\ &\dot v_n^\flat(\tau) \to \dot v(\tau) \text{ strongly in } L^2(\Omega)\,. \end{aligned}$$

We note that for n large enough the function $v_n^\flat$ is a solution of the wave equation on $\Omega \setminus H_n(t+\eta_n)$ for $t \in [\tau, t_1^n - \eta_n]$, with homogeneous Neumann boundary condition. By the continuous dependence of the solutions on the data [10, Theorem 4.1 and Example 4.3] we have

$$\begin{aligned} &v_n^\flat(t) \to v(t) \text{ strongly in } H^1(\Omega \setminus \Lambda)\,,\\ &\dot v_n^\flat(t) \to \dot v(t) \text{ strongly in } L^2(\Omega)\,, \end{aligned} \tag{6.15}$$

for every t such that $t \in [\tau, t_1^n - \eta_n]$ for n large enough. In particular, $t = t_\infty$ satisfies this condition by (6.12) and the definition of η_n. Since $v_n^\flat(t_\infty) = v_n(t_n)$, the strong convergence (6.13) follows from (6.15). □

We are now ready to conclude the proof of the main result of this section.

Proof of Theorem 6.1. By Remark 3.4 we obtain (6.5) for every $t \in [T_0, \tau_1)$.

It remains to consider the case $t = \tau_1$.

Note that every subsequence τ_1^n has a further subsequence, not relabelled, such that one of the following conditions is satisfied:

$$\tau_1 \le \tau_1^n \quad \text{for every } n\,, \tag{6.16}$$

$$\tau_1^n < \tau_1 \quad \text{for every } n\,. \tag{6.17}$$

In case (6.16) the result follows from Remark 3.4. In case (6.17) we necessarily have $\tau_1 = \tau_1^\infty$. Then we can apply Proposition 6.2 with $t_n := \tau_1^n$ and we obtain

$$\begin{aligned} &u_n(\tau_1^n) \to u(\tau_1) \text{ strongly in } H^1(\Omega \setminus \Gamma)\,,\\ &\dot u_n(\tau_1^n) \to \dot u(\tau_1) \text{ strongly in } L^2(\Omega)\,. \end{aligned} \tag{6.18}$$

We now apply Lemma 4.1 with $t_0 := \tau_1$, $u^0 := u(\tau_1)$, $u^1 := \dot u(\tau_1)$, $\tau_0 := \tau_1^n$, $\tau_\delta := (\tau_1^n + \delta) \wedge T_1$, $H(t) = K_n(t)$, $v^0 := u_n(\tau_1^n)$, $v^1 := \dot u_n(\tau_1^n)$, and $v(t) := u_n(t)$. Note that given $\delta > 0$ by (6.8) the inequalities $|\tau_1^n - \tau_1| < \delta$ and $\mathcal{E}(u_n(\tau_1^n) - u(\tau_1), \dot u_n(\tau_1^n) - \dot u(\tau_1), \Omega) < \delta$ are satisfied for n large enough. Since $(\sigma_n, u_n) \in \mathcal{A}_{reg}(T_0, T_1)$ the energy inequality for $v := u_n$ follows from Definition 3.2 and (2.9). Therefore, since $\tau_1 = \tau_1^\infty \in [\tau_1^n, (\tau_1^n + \delta) \wedge T_1]$ for n large enough, Lemma 4.1 implies that for every $\varepsilon > 0$

$$\mathcal{E}(u_n(\tau_1) - u(\tau_1), \dot u_n(\tau_1) - \dot u(\tau_1), \Omega) < \varepsilon$$

for n large enough. Since $\varepsilon > 0$ is arbitrary we have

$$\begin{aligned} &\nabla u_n(\tau_1) \to \nabla u(\tau_1) \quad \text{strongly in } L^2(\Omega \setminus \Gamma; \mathbb{R}^2)\,, \\ &\dot u_n(\tau_1) \to \dot u(\tau_1) \quad \text{strongly in } L^2(\Omega)\,. \end{aligned} \tag{6.19}$$

By integrating term by term the second line in (6.5) between T_0 and τ_1 we obtain, using the initial conditions, that $u_n(\tau_1) \to u(\tau_1)$ strongly in $L^2(\Omega)$. Together with (6.19) this completes the proof of (6.5) in the case $t = \tau_1$. □

7. Strong convergence while crossing the kink

Let σ_n, u_n, σ, K, and u be as in (5.1)-(5.4) and (6.1)-(6.4), and let

$$\tau_2 := \inf\{t \in [T_0, T_1] : \sigma(t) > 0\}\,, \tag{7.1}$$

with the convention $\inf \emptyset = T_1$. In this section we consider in detail the special case

$$\tau_1 < \tau_2\,. \tag{7.2}$$

Then we have by (2.4)

$$\sigma(t) = 0, \quad \text{hence} \quad K(t) = \Gamma_1 := \gamma([-L_1, 0]), \quad \text{for every } t \in [\tau_1, \tau_2]\,. \tag{7.3}$$

Our goal is to prove the following result.

Theorem 7.1. *Assume* (5.1)-(5.4), (6.1)-(6.4), *and* (7.1)-(7.3). *Then*

$$\begin{aligned} &u_n(t) \to u(t) \textit{ strongly in } H^1(\Omega \setminus \Gamma)\,, \\ &\dot u_n(t) \to \dot u(t) \textit{ strongly in } L^2(\Omega)\,, \end{aligned} \tag{7.4}$$

for every $t \in [T_0, \tau_2]$.

We shall use the following proposition which deals with the convergence in $[\tau_1, \tau_2]$.

Proposition 7.2. *Assume* (5.1)-(5.4), (6.1)-(6.4), *and* (7.1)-(7.3). *Then*

$$\sup_{t \in [\tau_1, \tau_2]} \mathcal{E}(u_n(t) - u(t), \dot u_n(t) - \dot u(t), \Omega) \to 0\,. \tag{7.5}$$

Proof. Let us first prove that

$$\int_{\tau_1}^{\tau_2} \mathcal{E}(u_n(t) - u(t), \dot u_n(t) - \dot u(t), \Omega) dt \to 0\,. \tag{7.6}$$

By (2.9) and (5.1) we have

$$\mathcal{E}(u_n(t), \dot u_n(t), \Omega) = \mathcal{E}((u_n(\tau_1), \dot u_n(\tau_1), \Omega) + \sigma_n(\tau_1) - \sigma_n(t) \tag{7.7}$$

for every $t \in [\tau_1, \tau_2]$. By (7.3) and by the energy equality for solutions on time-independent domains we have

$$\mathcal{E}(u(t), \dot u(t), \Omega) = \mathcal{E}(u(\tau_1), \dot u(\tau_1), \Omega) \tag{7.8}$$

for every $t \in [\tau_1, \tau_2]$. By (5.3) and (7.3) we have that

$$\sigma_n(\tau_1) - \sigma_n(t) \to \sigma(\tau_1) - \sigma(t) = 0 \tag{7.9}$$

uniformly for $t \in [\tau_1, \tau_2]$.

We claim that

$$\begin{aligned} &u_n(\tau_1) \to u(\tau_1) \quad \text{strongly in } H^1(\Omega \setminus \Gamma) \\ &\dot u_n(\tau_1) \to \dot u(\tau_1) \quad \text{strongly in } L^2(\Omega \setminus \Gamma)\,. \end{aligned} \tag{7.10}$$

If $T_0 < \tau_1$ the convergence was proved in Theorem 6.1. When $T_0 = \tau_1$, by (5.2) and (6.2) we have $u_n(\tau_1) = u(\tau_1) = u^0$ and $\dot u_n(\tau_1) = \dot u(\tau_1) = u^1$.

By (7.7)-(7.10) we conclude that

$$\mathcal{E}(u_n(t), \dot u_n(t), \Omega) \to \mathcal{E}(u(t), \dot u(t), \Omega) \tag{7.11}$$

uniformly for $t \in [\tau_1, \tau_2]$. This implies that

$$\int_{\tau_1}^{\tau_2} \mathcal{E}(u_n(t), \dot{u}_n(t), \Omega) dt \to \int_{\tau_1}^{\tau_2} \mathcal{E}(u(t), \dot{u}(t), \Omega) dt \,. \tag{7.12}$$

To prove (7.6) we observe that by (4.1) for a.e. $t \in (\tau_1, \tau_2)$ we have

$$\begin{aligned} &\mathcal{E}(u_n(t) - u(t), \dot{u}_n(t) - \dot{u}(t), \Omega) \\ &= \mathcal{E}(u_n(t), \dot{u}_n(t), \Omega) - (\nabla u_n(t), \nabla u(t)) - (\dot{u}_n(t), \dot{u}(t)) + \mathcal{E}(u(t), \dot{u}(t), \Omega) \,. \end{aligned} \tag{7.13}$$

Integrating this equality from τ_1 to τ_2 we obtain

$$\begin{aligned} &\int_{\tau_1}^{\tau_2} \mathcal{E}(u_n(t) - u(t), \dot{u}_n(t) - \dot{u}(t), \Omega) dt = \int_{\tau_1}^{\tau_2} \mathcal{E}(u_n(t), \dot{u}_n(t), \Omega) dt \\ &- \int_{\tau_1}^{\tau_2} (\nabla u_n(t), \nabla u(t)) dt - \int_{\tau_1}^{\tau_2} (\dot{u}_n(t), \dot{u}(t)) dt + \int_{\tau_1}^{\tau_2} \mathcal{E}(u(t), \dot{u}(t), \Omega) dt \,, \end{aligned} \tag{7.14}$$

which, together with (7.12) and the weak convergence (5.5), gives (7.6). Therefore a subsequence, not relabelled, satisfies

$$\mathcal{E}(u_n(t) - u(t), \dot{u}_n(t) - \dot{u}(t), \Omega) \to 0 \,, \tag{7.15}$$

for a.e. $t \in [\tau_1, \tau_2]$.

To prove (7.5) we begin by observing that

$$u \in C^0([\tau_1, \tau_2]; H^1(\Omega \setminus \Gamma_1)) \quad \text{and} \quad \dot{u} \in C^0([\tau_1, \tau_2]; L^2(\Omega)) \,.$$

This is a consequence of (7.3), of Theorem 5.1, and of the continuity of the solutions of the wave equation in time-independent domains (see, e.g., [17, Thm. 8.2]). By uniform continuity for every $\varepsilon > 0$ there exists $\omega(\varepsilon) > 0$ such that if $t_1, t_2 \in [\tau_1, \tau_2]$ with $|t_2 - t_1| < \omega(\varepsilon)$ then

$$\mathcal{E}(u(t_2) - u(t_1), \dot{u}(t_1) - \dot{u}(t_2), \Omega)) < \varepsilon \,.$$

Choose $t_0 \in [\tau_1, \tau_2]$. We now apply Lemma 4.1 with $u^0 := u(t_0)$ and $u^1 := \dot{u}(t_0)$. Given ε, there exists $0 < \delta < \omega(\varepsilon^2)$ such that (a)-(d) imply (4.3). We now choose η such that $0 < \eta < \delta \wedge \omega(\delta)$. If $t_0 > \tau_1$ we choose $\tau_0 \in ((t_0 - \eta) \vee \tau_1, t_0)$ such that (7.15) holds for $t = \tau_0$. If $t_0 = \tau_1$ we choose $\tau_0 = t_0 = \tau_1$ and observe that (7.15) holds for τ_0 by (7.10). In both cases we set $\tau_\delta := (\tau_0 + \delta) \wedge T_1$. Finally, we choose $v^0 := u_n(\tau_0)$, $v^1 := \dot{u}_n(\tau_0)$, and $v(t) := u_n(t)$.

We check that conditions (a)-(d) in Lemma 4.1 are satisfied for n large enough. Condition (a) is trivial since $\eta < \delta$. Condition (b) requires that

$$\mathcal{E}(u_n(\tau_0) - u(t_0), \dot{u}_n(\tau_0) - \dot{u}(t_0), \Omega) < \delta \tag{7.16}$$

for n large enough. By the choice of η we have

$$\mathcal{E}(u(\tau_0) - u(t_0), \dot{u}(\tau_0) - \dot{u}(t_0), \Omega) < \delta \,,$$

which implies (7.16) by the pointwise convergence (7.15) for $t = \tau_0$. Condition (c) is trivial and (d) follows from the energy-dissipation balance (2.9) for (σ_n, u_n).

Therefore by Lemma 4.1 there exists n_{t_0} such that

$$\sup_{t \in [\tau_0, \tau_\delta]} \mathcal{E}(u_n(t) - u(t_0), \dot{u}_n(t) - \dot{u}(t_0), \Omega) \le \varepsilon$$

for every $n \ge n_{t_0}$.

Since $\delta < \omega(\varepsilon^2)$ and every $t \in [\tau_0, \tau_\delta]$ satisfies $|t_0 - t| < \delta$, we obtain

$$\sup_{t \in [\tau_0, \tau_\delta]} \mathcal{E}(u(t) - u(t_0), \dot{u}(t) - \dot{u}(t_0), \Omega) \le \varepsilon \,,$$

and by the triangle inequality for $\sqrt{\mathcal{E}}$ we get

$$\sup_{t \in [\tau_0, \tau_\delta]} \mathcal{E}(u_n(t) - u(t), \dot{u}_n(t) - \dot{u}(t), \Omega) \le 4\varepsilon \tag{7.17}$$

for every $n \ge n_{t_0}$.

Suppose that $t_0 > \tau_1$. Let us check that in this case $\tau_0 < t_0 < \tau_0 + \delta$. The first inequality follows from our choice of τ_0; the second one is a consequence of the inequalities $t_0 - \eta < \tau_0$ and $\eta < \delta$, which give $t_0 < t_0 - \eta + \delta < \tau_0 + \delta$. Therefore $I_{t_0} := (\tau_0, \tau_0 + \delta)$ is an open neighbourhood of t_0 and by (7.17) we have

$$\sup_{t \in I_{t_0} \cap [\tau_1, \tau_2]} \mathcal{E}(u_n(t) - u(t), \dot{u}_n(t) - \dot{u}(t), \Omega) \le 4\varepsilon \tag{7.18}$$

for every $n \geq n_{t_0}$.

If $t_0 = \tau_1$ the interval $I_{t_0} := (\tau_1 - \delta, \tau_1 + \delta)$ is an open neighbourhood of t_0 and by (7.17) we have (7.18) also in this case.

Since $[\tau_1, \tau_2]$ is contained in the union of the open sets I_{t_0} for $t_0 \in [\tau_1, \tau_2]$, there exist $t_1, \dots, t_k$ such that $[\tau_1, \tau_2] \subset I_{t_1} \cup \dots \cup I_{t_k}$. By (7.18) we obtain that for every $\varepsilon > 0$ there exists n_ε such that

$$\sup_{t \in [\tau_1, \tau_2]} \mathcal{E}(u_n(t) - u(t), \dot{u}_n(t) - \dot{u}(t), \Omega) \leq 4\varepsilon$$

for $n \geq n_\varepsilon$, which gives (7.5) for the subsequence satisfying (7.15). Since the limit does not depend on the subsequence, (7.5) holds for the entire sequence. □

Proof of Theorem 7.1. If $T_0 < \tau_1$, by Theorem 6.1 we have that (7.4) holds for every $t \in [T_0, \tau_1]$. It remains to prove (7.4) for $t \in [\tau_1, \tau_2]$ when $T_0 \leq \tau_1 < \tau_2$. By Proposition 7.2 we have

$$\begin{aligned} &\nabla u_n(t) \to \nabla u(t) \quad \text{strongly in } L^2(\Omega \setminus \Gamma; \mathbb{R}^2), \\ &\dot{u}_n(t) \to \dot{u}(t) \quad \text{strongly in } L^2(\Omega), \end{aligned} \tag{7.19}$$

uniformly for $t \in [\tau_1, \tau_2]$. Since $u_n(\tau_1) \to u(\tau_1)$ by (7.10), from (7.19) we obtain by integration that

$$u_n(t) \to u(t) \quad \text{strongly in } L^2(\Omega) \tag{7.20}$$

uniformly for $t \in [\tau_1, \tau_2]$. From (7.19) and (7.20) we obtain (7.4) uniformly for $t \in [\tau_1, \tau_2]$. □

8. Strong convergence leaving the kink

In this section we complete the proof of Theorem 3.3 by studying the convergence for $t \in (\tau_2, T_1]$.

Proof of Theorem 3.3. Let us prove that (3.3) holds for every $t \in [T_0, \tau_2]$. If $\tau_1 < \tau_2$ this is proved in Theorem 7.1. If $T_0 < \tau_1 = \tau_2$ the result follows from Theorem 6.1. In the remaining case $T_0 = \tau_1 = \tau_2$, we use (5.2) and (6.2).

It remains to prove (3.3) for $t \in (\tau_2, T_1]$. If $\sigma(\tau_2) > 0$ the result follows from Remark 3.4.

Let us now consider the critical case $\sigma(\tau_2) = 0$. We apply Lemma 4.1 with $t_0 := \tau_2$, $u^0 := u(\tau_2)$, $u^1 := \dot{u}(\tau_2)$, $\tau_0 := \tau_2$, $v^0 := u_n(\tau_2)$, $v^1 := \dot{u}_n(\tau_2)$, and $v(t) := u_n(t)$. By (3.3) for $t = \tau_2$ condition (b) of Lemma 4.1 holds for n large enough. Condition (c) is trivial, while (d) follows from (2.9). Therefore, given $\varepsilon > 0$ there exist $0 < \delta < \varepsilon \wedge (T_1 - \tau_2)$ and n_1 such that

$$\mathcal{E}(u_n(t) - u(\tau_2), \dot{u}_n(t) - \dot{u}(\tau_2), \Omega) < \varepsilon \tag{8.1}$$

for every $n \geq n_1$ and for every $t \in [\tau_2, \tau_2 + \delta]$. For every $[a, b] \subset [\tau_2, \tau_2 + \delta]$ this implies

$$\frac{1}{b-a} \int_a^b \mathcal{E}(u_n(t) - u(\tau_2), \dot{u}_n(t) - \dot{u}(\tau_2), \Omega) dt \leq \varepsilon .$$

Since $u_n \rightharpoonup u$ weakly in $L^2((T_0, T_1); H^1(\Omega \setminus \Gamma))$ and $\dot{u}_n \rightharpoonup \dot{u}$ weakly in $L^2((T_0, T_1); L^2(\Omega))$, by lower semicontinuity we have that

$$\frac{1}{b-a} \int_a^b \mathcal{E}(u(t) - u(\tau_2), \dot{u}(t) - \dot{u}(\tau_2), \Omega) dt \leq \varepsilon .$$

This implies that

$$\mathcal{E}(u(t) - u(\tau_2), \dot{u}(t) - \dot{u}(\tau_2), \Omega) \leq \varepsilon \tag{8.2}$$

for a.e. $t \in [\tau_2, \tau_2 + \delta]$. By (5.6) this inequality holds true for every $t \in [\tau_2, \tau_2 + \delta]$.

We fix $\hat{t} \in (\tau_2, \tau_2 + \delta)$. Since $\sqrt{\mathcal{E}}$ satisfies the triangle inequality, by (8.1) and (8.2) we obtain

$$\mathcal{E}(u_n(\hat{t}) - u(\hat{t}), \dot{u}_n(\hat{t}) - \dot{u}(\hat{t}), \Omega) < 4\varepsilon \tag{8.3}$$

for every $n \geq n_1$.

Since $\sigma(\hat{t}) > 0$ we can construct diffeomorphisms $\Phi^n : \Omega \setminus K_n(\hat{t}) \to \Omega \setminus K(\hat{t})$ such that $\Phi^n \to id$ in C^2. This implies that

$$\mathcal{E}(u(\hat{t}, \Phi^n) - u(\hat{t}), \dot{u}(\hat{t}, \Phi^n) - \dot{u}(\hat{t}), \Omega) \to 0 . \tag{8.4}$$

Therefore there exists $n_2 \geq n_1$ such that

$$\mathcal{E}(u(\hat{t}, \Phi^n) - u(\hat{t}), \dot{u}(\hat{t}, \Phi^n) - \dot{u}(\hat{t}), \Omega) < \varepsilon \tag{8.5}$$

for every $n \geq n_2$.

Let now v_n be the solution of the wave equation on the time-dependent cracking domains $\Omega\backslash K_n(t)$, $t \in [\hat{t}, T_1]$, with initial conditions $v_n(\hat{t}) = u(\hat{t}, \Phi^n)$ and $\dot{v}_n(\hat{t}) = \dot{u}(\hat{t}, \Phi^n)$ and homogeneous Neumann boundary conditions. Since $\sigma(T_1) < L_2$, the regularity assumptions on σ and σ_n allow us to apply the continuous dependence of the solutions on the cracks (in the case of varying initial cracks) and by (8.4) we have (see [9, Theorem 3.5])

$$v_n(t) \to u(t) \quad \text{strongly in } H^1(\Omega \setminus \Gamma) \quad \text{and} \quad \dot{v}_n(t) \to \dot{u}(t) \quad \text{strongly in } L^2(\Omega) \tag{8.6}$$

for every $t \in [\hat{t}, T_1]$.

We write

$$u_n = v_n + z_n\,. \tag{8.7}$$

Then z_n is the solution of the wave equation on the time-dependent cracking domains $\Omega \setminus K_n(t)$, $t \in [\hat{t}, T_1]$, with initial conditions $z_n(\hat{t}) = u_n(\hat{t}) - u(\hat{t}, \Phi^n)$ and $\dot{z}_n(\hat{t}) = \dot{u}_n(\hat{t}) - \dot{u}(\hat{t}, \Phi^n)$, and homogeneous Neumann boundary condition. Since $\sigma_n(t) \geq \sigma_n(\hat{t}) > 0$ for every $t \in [\hat{t}, T_1]$ and for n large enough, we can construct the t-dependent diffeomorphisms considered in [10], therefore the solution z_n is unique (see [10, Corollary 3.3]). By the energy inequality (due to uniqueness on $[\hat{t}, T_1]$), see [11, Corollary 3.2], we have

$$\mathcal{E}(z_n(t), \dot{z}_n(t), \Omega) \leq \mathcal{E}(z_n(\hat{t}), \dot{z}_n(\hat{t}), \Omega)$$

for every $t \in [\hat{t}, T_1]$. Hence by (8.3) and (8.5), using the triangle inequality for $\sqrt{\mathcal{E}}$ we obtain

$$\mathcal{E}(z_n(t), \dot{z}_n(t), \Omega) \leq 9\varepsilon \tag{8.8}$$

for every $t \in [\hat{t}, T_1]$ and $n \geq n_2$.

Therefore by (8.7) and (8.8) we obtain

$$\begin{aligned}
&\mathcal{E}(u_n(t) - u(t), \dot{u}_n(t) - \dot{u}(t), \Omega)^{1/2} \\
&\leq \mathcal{E}(v_n(t) - u(t), \dot{v}_n(t) - \dot{u}(t), \Omega)^{1/2} + \mathcal{E}(z_n(t), \dot{z}_n(t), \Omega)^{1/2} \\
&\leq \mathcal{E}(v_n(t) - u(t), \dot{v}_n(t) - \dot{u}(t), \Omega)^{1/2} + 3\varepsilon^{1/2}
\end{aligned}$$

for every $t \in [\hat{t}, T_1]$ and $n \geq n_2$. By (8.6), recalling that $\hat{t} \leq \tau_2 + \delta \leq \tau_2 + \varepsilon$, we have that for every $t \in [\tau_2 + \varepsilon, T_1]$ there exists $n_3(t) \geq n_2$ such that

$$\mathcal{E}(u_n(t) - u(t), \dot{u}_n(t) - \dot{u}(t), \Omega)^{1/2} \leq 4\varepsilon^{1/2}$$

for every $n \geq n_3(t)$. This implies that for every $t \in (\tau_2, T_1]$

$$\mathcal{E}(u_n(t) - u(t), \dot{u}_n(t) - \dot{u}(t), \Omega) \to 0\,,$$

hence

$$\begin{aligned}
&\nabla u_n(t) \to \nabla u(t) \quad \text{strongly in } L^2(\Omega \setminus \Gamma; \mathbb{R}^2)\,, \\
&\dot{u}_n(t) \to \dot{u}(t) \quad \text{strongly in } L^2(\Omega)\,,
\end{aligned} \tag{8.9}$$

for $t \in (\tau_2, T_1]$. Since (3.3) for $t = \tau_2$ gives in particular that $u_n(\tau_2) \to u(\tau_2)$ strongly in $L^2(\Omega)$, from the second line of (8.9) we obtain by integration that

$$u_n(t) \to u(t) \quad \text{strongly in } L^2(\Omega) \tag{8.10}$$

for $t \in (\tau_2, T_1]$. From (8.9) and (8.10) we obtain (3.3) for $t \in (\tau_2, T_1]$. □

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References

[1] Adda-Bedia M., Arias R.: *Brittle fracture dynamics with arbitrary paths I. Kinking of dynamic crack in general antiplane loading.* J. Mech. Phys. Solids **51** (2003), 1287-1304.

[2] Achenbach J.D., Kuo M.K., Dempsey J.P.: *Mode III and mixed mode I-II crack kinking under stress-wave loading.* Internat. J. Solids Structures **20** (1984), 395-410.

[3] Broberg K.B.: *Cracks and Fracture.* Academic Press, London, 1999.

[4] Burgers P.: *Dynamic propagation of a kinked or bifurcated crack in antiplane strain.* J. Appl. Mech. **49** (1983), 37-376.

[5] Burgers P.: *Dynamic kinking of a crack in plane strain.* Internat. J. Solids Structures **19** (1984), 735-752.

[6] Caponi M.: *Linear hyperbolic systems in domains with growing cracks.* Milan J. Math. **85** (2017), 149-185.

[7] Dal Maso G., Larsen C.J.: *Existence for wave equations on domains with arbitrary growing cracks.* Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **22** (2011), 387-408.

[8] Dal Maso G., Larsen C.J., Toader R.: *Existence for constrained dynamic Griffith fracture with a weak maximal dissipation condition.* J. Mech. Phys. Solids **95** (2016), 697-707.

[9] Dal Maso G., Larsen C.J., Toader R.: *Existence for dynamic Griffith fracture with a weak maximal dissipation condition.* J. Math. Pures. Appl. (9) **127** (2019), 160-191.

[10] Dal Maso G., Lucardesi I.: *The wave equation on domains with cracks growing on a prescribed path: existence, uniqueness, and continuous dependence on data.* Appl. Math. Res. Express. AMRX.

[11] Dal Maso G., Toader R.: *On the Cauchy problem for the wave equation on time-dependent domains.* J. Differential Equations **266** (2019), 3209–3246.

[12] Dempsey J.P., Kuo M.K., Achenbach J.D.: *Mode III kinking under stress wave loading.* Wave Motion **4**, (1982), 181-190.

[13] Freund L.B.: *Dynamic Fracture Mechanics.* Cambridge Univ. Press, New York, 1990.

[14] Griffith A.: *The phenomena of rupture and flow in solids.* Philos. Trans. Roy. Soc. London Ser. A **221** (1920), 163-198.

[15] Lawn B.: *Fracture of Brittle Solids.* Cambridge University Press, New York, 1993

[16] Larsen C.J.: *Models for dynamic fracture based on Griffiths criterion,* in IUTAM Symp. Variational Concepts with Applications to the Mechanics of Materials, ed. K. Hackl (Springer, 2010), pp. 131-140.

[17] Lions J.-L., Magenes E.: *Non-Homogeneous Boundary Value Problems and Applications. I.* Translated from the French by P. Kenneth. Die Grundlehren der mathematischen Wissenschaften, Band 181. Springer-Verlag, New York-Heidelberg, 1972.

[18] Mott N.F.: *Brittle fracture in mild steel plates.* Engineering **165** (1948), 16-18.

[19] Theocaris P. S., Makrakis G.: *Crack kinking in anti-plane shear solved by the Mellin transform.* Internat. J. Fracture **34** (1987), 251-262.

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